ASSIGNMENT-03

Rank and System of linear equations

Last Date of Submission: 12-07-2022, 23:59 Hours, Tuesday (in the Google classroom)

Rank of a Matrix

Suppose that $A_{m\times n}$ is reduced by row operations to an echelon form **E.** The rank of matrix A is defined to be the number

$$\begin{aligned}\text{rank}(A) &= \text{number of pivots}\\ &= \text{number of non-zero rows in E}\\ &= \text{number of basic columns in A}\end{aligned}$$

Where **basic columns of A** are defined to be those columns in A that contains the pivotal positions.

01. Reduce each of the following matrices into row echelon form, determine the rank and identify the basic columns.

(a) $\begin{pmatrix} 1 & 2 & 3 & 3\\ 2 & 4 & 6 & 9\\ 2 & 6 & 7 & 6\end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3\\ 2 & 6 & 8\\ 2 & 6 & 0\\ 1 & 2 & 5\\ 3 & 8 & 6\end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1\\ 4 & 2 & 4 & 4 & 1 & 5 & 5\\ 2 & 1 & 3 & 1 & 0 & 4 & 3\\ 6 & 3 & 4 & 8 & 1 & 9 & 5\\ 0 & 0 & 3 & -3 & 0 & 0 & 3\\ 8 & 4 & 2 & 14 & 1 & 13 & 3\end{pmatrix}$

02. Suppose that A is an $m\times n$ matrix. Give a short explanation of why each of the following statement is true?

(a) rank(A) $\le$ min{m, n} .

(b) rank(A) < m, if one row in A is entirely zero.

(c) rank(A) < m, if one row in A is a multiple of another row.

(d) rank(A) < m, if one row in A is a combination of other rows.

(e) rank(A) < n, if one column in A is entirely zero.

03. Determine which of the following systems are consistent and solve them.

(a) $\begin{aligned} x+2y+z&=2 \\ 2x+4y&=2 \\ 3x+6y+z&=4 \end{aligned}$ (b) $\begin{aligned} 2x+2y+4z&=0 \\ 3x+2y+5z&=0 \\ 4x+2y+6z&=0 \end{aligned}$ (c) $\begin{aligned} x-y+z&=1 \\ x-y-z&=2 \\ x+y-z&=3 \\ x+y+z&=4 \end{aligned}$

(d) $\begin{aligned} x-y+z&=1 \\ x-y-z&=2 \\ x+y-z&=3 \\ x+y+z&=2 \end{aligned}$ (e) $\begin{aligned} 2w+x+3y+5z&=1 \\ 4w+4y+8z&=0 \\ w+x+2y+3z&=0 \\ x+y+z&=0 \end{aligned}$ (f) $\begin{aligned} 2w+x+3y+5z&=7 \\ 4w+4y+8z&=8 \\ w+x+2y+3z&=5 \\ x+y+z&=3 \end{aligned}$

04. If A is an $m\times n$ matrix with rank(A) = m, explain why the system [A|b] must be consistent for every right hand side b?

05. Explain why the following homogeneous system has only trivial solutions.

$$\begin{aligned} x_1+2x_2+2x_3&=0 \\ 2x_1+5x_2+7x_3&=0 \\ 3x_1+6x_2+8x_3&=0 \end{aligned}$$

06. Explain why the following homogeneous system has infinitely many solutions, and exhibit the general solution.

$$\begin{aligned} x_1+2x_2+2x_3&=0 \\ 2x_1+5x_2+7x_3&=0 \\ 3x_1+6x_2+6x_3&=0 \end{aligned}$$

07. If A is the coefficient matrix for a homogeneous system consisting of four equations in eight unknowns and if there are five free variables, what is rank(A)?

08. Suppose that A is the coefficient matrix for a homogeneous system of four equations in six unknowns and suppose that A has at least one non-zero row,

(a) Determine the fewest number of free variables that are possible.

(b) Determine the maximum number of free variables that are possible.

09. Explain why a homogeneous system of m equations in n unknowns where m < n must always posses an infinite number of solutions.

10. Among the solutions that satisfy the set of linear equations

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2 \\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3 \end{aligned}$$

find all those that satisfy the following two constraints:

$$\begin{aligned} (x_1 - x_2)^2 - 4x_5^2 &= 0 \\ x_3^2 - x_5^2 &= 0 \end{aligned}$$

11. Consider the following system:

$$\begin{aligned} 2x + 2y + 3z &= 0 \\ 4x + 8y + 12z &= -4 \\ 6x + 2y + \alpha z &= 4 \end{aligned}$$

(a) Determine all the values of α for which the system is consistent.

(b) Determine the value of α for which there is a unique solution, and compute the solution for these cases.

(c) Determine all values of α for which there are infinitely many different solutions, and give the general solution for these cases.

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