LOGARITHMIC & EXPONENTIAL FUNCTION

Definition of the Natural Logarithm Function

The natural logarithm is the function given by $\ln x = \int_1^x \frac{1}{t}\,dt \quad , x > 0$.

Theorem-01: For any numbers $b > 0$ and $x > 0$, prove that

(a) $\ln(bx) = \ln b + \ln x$

(b) $\ln\left(\frac{b}{x}\right) = \ln b - \ln x$

(c) $\ln\left(\frac{1}{x}\right) = -\ln x$

(d) $\ln\left(x^r\right) = r\ln x$.

Definition of the Exponential Function

For every real number *x*, we define the **natural exponential function** to be $e^x = \exp(x)$.

Inverse Equations of e^x and $\ln x$

$$e^{\ln x} = x, \qquad \text{for } x > 0$$
$$\ln\left(e^x\right) = x, \qquad \text{for all x}$$

Example-01: Solve the equation $e^{2x-6} = 4$ for x. **Ans:** $x = 3 + \ln 2$.

Example-02: A line with slope *m* passes through the origin and is tangent to the graph of *y* = ln *x*. What is the value of *m*? **Ans:** $m = \frac{1}{e}$.

Theorem-02(Laws of Exponent): For all x, x_1, x_2, the exponential function e^x obeys the following laws:

(a) $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

(b) $e^{-x} = \frac{1}{e^x}$

(c) $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

(d) $\left(e^x\right)^r = e^{rx}$, if r is real

General Exponential Function

For any number $a>0$ and any real x, the exponential function with base a is defined as $a^x = e^{x\ln a}$.

Theorem-03(General Laws of Exponent): For all x, x_1, x_2, the exponential function a^x obeys the following laws:

(a) $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$

(b) $a^{-x} = \dfrac{1}{a^x}$

(c) $\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}$

(d) $\left(a^x\right)^r = a^{rx}$, if r is real

Definition: For any $x>0$ and for any real number n, we have $x^n = e^{n\ln x}$.

Example-03: Differentiate $f(x) = x^x, x>0$. **Ans:** $x^x(\ln x + 1)$

Theorem-04(The Number *e* as a Limit) The number *e* can be calculated as the limit $e = \lim\limits_{x\to 0}(1+x)^{1/x}$.

Logarithm with base a

For any positive number $a \neq 1$, $\log_a x$ is the inverse of a^x. i.e. $y = a^x \Rightarrow x = \log_a y$.

Theorem-05: For any numbers $x>0$ and $y>0$, we have

(a) $\log_a x$ is just a numerical multiple of $\ln x$, i.e. $\log_a x = \dfrac{\ln x}{\ln a}$.

(b) $\log_a(xy) = \log_a x + \log_a y$

(c) $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

(d) $\log_a\left(\dfrac{1}{y}\right) = -\log_a y$

(e) $\log_a\left(x^y\right) = y\log_a x$.

Example-04: Evaluate (a) $\dfrac{d}{dx}\log_{10}(3x+1)$ (b) $\int \dfrac{\log_2 x}{x}dx$ **Ans:** (a) $\dfrac{3}{(\ln 10)(3x+1)}$, (b) $\dfrac{(\ln x)^2}{2\ln 2} + c$.

1. Evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta}d\theta$.

2. Find $\dfrac{dy}{dx}$, if $y = \dfrac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}, x>1$.

Evaluating Integrals

Evaluate the integrals in Exercises 3-20.

3. $\int_{-3}^{-2} \frac{dx}{x}$
4. $\int_{-1}^{0} \frac{3\,dx}{3x-2}$
5. $\int_{2}^{4} \frac{dx}{x(\ln x)^2}$
6. $\int_{2}^{16} \frac{dx}{2x\sqrt{\ln x}}$
7. $\int \frac{2y\,dy}{y^2-25}$
8. $\int \frac{8r\,dr}{4r^2-5}$
9. $\int \frac{3\sec^2 t}{6+3\tan t}dt$
10. $\int \frac{\sec y \tan y}{2+\sec y}dy$
11. $\int_{0}^{\pi} \frac{\sin t}{2-\cos t}dt$
12. $\int_{0}^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}d\theta$
13. $\int_{0}^{\pi/2} \tan\frac{x}{2}dx$
14. $\int_{\pi/4}^{\pi/2} \cot t\,dt$
15. $\int_{1}^{2} \frac{2\ln x}{x}dx$
16. $\int_{2}^{4} \frac{dx}{x\ln x}$
17. $\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}d\theta$
18. $\int_{0}^{\pi/12} 6\tan 3x\,dx$
19. $\int \frac{dx}{2\sqrt{x}+2x}$
20. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$

Logarithmic Differentiation

In Exercises 21-34, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

21. $y=\sqrt{x(x+1)}$
22. $y=\sqrt{\frac{t}{t+1}}$
23. $y=\sqrt{\theta+3}\sin\theta$
24. $y=t(t+1)(t+2)$
25. $y=\frac{\theta+5}{\theta\cos\theta}$
26. $y=\sqrt{(x^2+1)(x-1)^2}$
27. $y=\sqrt{\frac{1}{t(t+1)}}$
28. $y=(\tan\theta)\sqrt{2\theta+1}$
29. $y=\frac{1}{t(t+1)(t+2)}$
30. $y=\frac{\theta\sin\theta}{\sqrt{\sec\theta}}$
31. $y=\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$
32. $y=\sqrt[3]{\frac{x(x-2)}{x^2+1}}$
33. $y=\sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$
34. $y=\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

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