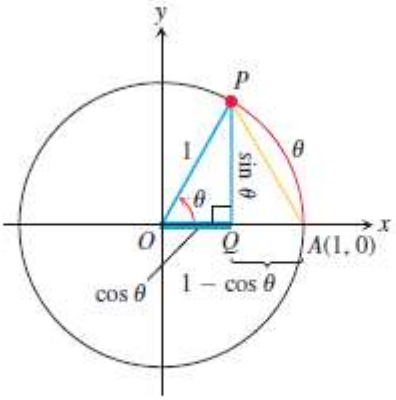
TRIGONOMETRIC & INVERSE TRIGONOMETRIC FUNCTION

Question: For any angle θ, measured in radians, prove that $-|\theta| \le \sin\theta \le |\theta|$ and $-|\theta| \le 1-\cos\theta \le |\theta|$.

Proof:

To establish these inequalities, we picture θ as a nonzero angle in standard position (see the figure).



The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP.

The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2. \qquad (1)$$

The terms on the left-hand side of Equation (1) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2:

$$\sin^2\theta \le \theta^2 \quad \text{and} \quad (1 - \cos\theta)^2 \le \theta^2.$$

By taking square roots, this is equivalent to saying that

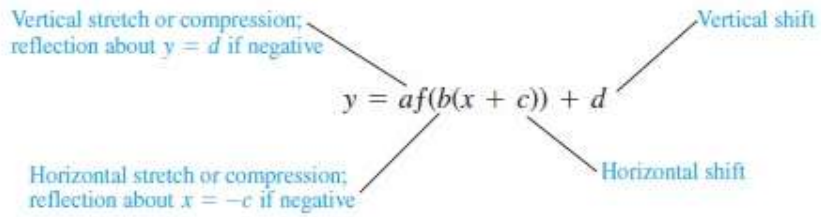
$$|\sin\theta| \le |\theta| \quad \text{and} \quad |1 - \cos\theta| \le |\theta|,$$

so

$$-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

Transformations of Trigonometric Graphs

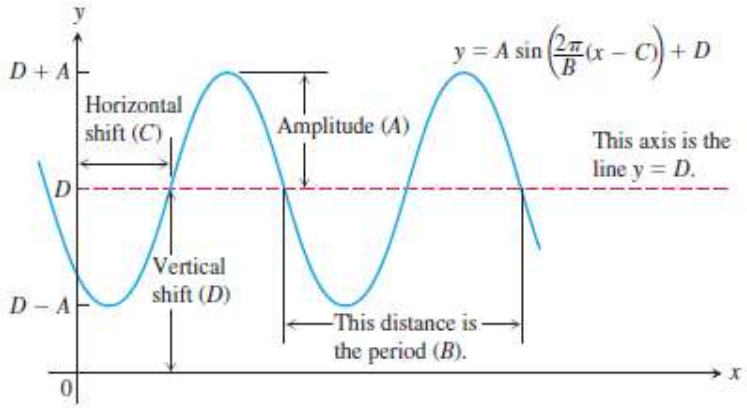
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions.



The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is given below.



Question: Prove the following:

(a) $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c, \quad \left(\text{valid for } u^2 < a^2\right).$

(b) $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c, \quad \left(\text{valid for all } u\right).$

(c) $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c, \quad \left(\text{valid for } |u| > a > 0\right).$

Question: Evaluate

(a) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}}$

(b) $\int \frac{dx}{\sqrt{3 - 4x^2}}$

(c) $\int \frac{dx}{\sqrt{e^{2x} - 6}}$

(d) $\int \frac{dx}{\sqrt{4x - x^2}}$

(e) $\int \frac{dx}{4x^2 + 4x + 2}$

EXERCISE

Evaluate the integrals in Exercises 1-48.

1. $\int \frac{dx}{\sqrt{9-x^2}}$
2. $\int \frac{dx}{\sqrt{1-4x^2}}$
3. $\int \frac{3\,dr}{\sqrt{1-4(r-1)^2}}$
4. $\int \frac{6\,dr}{\sqrt{4-(r+1)^2}}$
5. $\int \frac{dx}{17+x^2}$
6. $\int \frac{dx}{9+3x^2}$
7. $\int \frac{dx}{2+(x-1)^2}$
8. $\int \frac{dx}{1+(3x+1)^2}$
9. $\int \frac{dx}{x\sqrt{25x^2-2}}$
10. $\int \frac{dx}{x\sqrt{5x^2-4}}$
11. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$
12. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1+(\sin\theta)^2}$
13. $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}}$
14. $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}}$
15. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2-25}}$
16. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1+(\cot x)^2}$
17. $\int_0^2 \frac{dt}{8+2t^2}$
18. $\int_{-2}^{2} \frac{dt}{4+3t^2}$
19. $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}}$
20. $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1+\ln^2 t)}$
21. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}}$
22. $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$
23. $\int \frac{y\,dy}{\sqrt{1-y^4}}$
24. $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}}$
25. $\int \frac{dx}{\sqrt{-x^2+4x-3}}$
26. $\int \frac{dx}{\sqrt{2x-x^2}}$
27. $\int_1^2 \frac{8\,dx}{x^2-2x+2}$
28. $\int_2^4 \frac{2\,dx}{x^2-6x+10}$
29. $\int_{-1}^{0} \frac{6\,dt}{\sqrt{3-2t-t^2}}$
30. $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}}$
31. $\int \frac{x+4}{x^2+4}dx$
32. $\int \frac{t-2}{t^2-6t+10}dt$
33. $\int \frac{dy}{y^2-2y+5}$
34. $\int \frac{dy}{y^2+6y+10}$
35. $\int \frac{x^2+2x-1}{x^2+9}dx$
36. $\int \frac{t^3-2t^2+3t-4}{t^2+1}dt$
37. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$
38. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$
39. $\int \frac{dy}{(\tan^{-1}y)(1+y^2)}$
40. $\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}}$
41. $\int \frac{e^{\sin^{-1}x}\,dx}{\sqrt{1-x^2}}$
42. $\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}}$
43. $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$
44. $\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$
45. $\int \frac{(\sin^{-1}x)^2\,dx}{\sqrt{1-x^2}}$
46. $\int \frac{\sqrt{\tan^{-1}x}\,dx}{1+x^2}$
47. $\int \frac{1}{\sqrt{x}(x+1)\left((\tan^{-1}\sqrt{x})^2+9\right)}dx$
48. $\int \frac{e^x\sin^{-1}e^x}{\sqrt{1-e^{2x}}}dx$

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