BARNAGAR COLLEGE, SORBHOG

DEPARTMENT OF MATHEMATICS

SUMMER BREAK ASSIGNMENT-2023

01. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

02. Show that the vectors $e_1 = (1,0,0,...,0), e_2 = (0,1,0,...,0),...,e_n = (0,0,0,...,1) \in \mathbb{R}^n(\mathbb{R})$ are linearly independent.

03. If the set $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of the vector space V(F) is linearly independent, then show that none of these vectors can be zero vector.

04. Prove that every superset of linearly dependent set of vectors is linearly dependent.

05. Prove that every subset of linearly independent set of vectors is linearly independent.

06. Show that a system consisting of a single non-zero vector is always linearly independent.

07. Describe geometrically the linear dependence of any two vectors u and v in the vector space $\mathbb{R}^3$.

08. Let $U = \{(a,b,c) : a = b = c\}$ is a subset in $\mathbb{R}^3$. Determine whether or not U is a subspace of $\mathbb{R}^3$.

09. The set $\mathbb{R}$ of all real numbers is a vector space over the field $\mathbb{Q}$ of rational numbers. Examine whether or not the set $\{1, \sqrt{2}\}$ of vectors in $\mathbb{R}$ is linearly independent.

10. Express $v = (2,-5,3)$ in $\mathbb{R}^3$ as a linear combination of the vectors

$u_1 = (1,-3,2), u_2 = (2,-4,-1), u_3 = (1,-5,7)$.

11. Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,2,3), \beta = (3,1,0)$. Check if the vector $(2,1,3)$ is in this subspace.

12. Examine if the following set S is a subspace of $\mathbb{R}^3$: $S = \{(x,y,z) \in \mathbb{R}^3 : x + 2y - z = 0, 2x - y + z = 0\}$.

13. Prove that the set of vectors $\{(1,2,2),(2,1,2),(2,2,1)\}$ is linearly independent in $\mathbb{R}^3$.

14. Let S be the subset of $\mathbb{R}^3$ defined by $S=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=z^2\right\}$. Examine whether S is a subspace of $\mathbb{R}^3$.

15. In $\mathbb{R}^3, \alpha=(4,3,5), \beta=(0,1,3), \gamma=(2,1,1)$. Is α a linear combination of β and γ?

16. If F is a field of real numbers, prove that the vectors $(a_1,a_2),(b_1,b_2)\in\mathbb{R}^2(F)$ are linearly dependent if and only if $a_1b_2-a_2b_1=0$.

17. If α,β,γ are linearly independent vectors of V(F), where F is any subfield of the field of complex numbers, then show that the vectors $\alpha+\beta,\beta+\gamma,\gamma+\alpha$ are also linearly independent.

18. In the vector space $\mathbb{R}^3$, let $\alpha=(1,2,1), \beta=(3,1,5), \gamma=(3,-4,7)$. Show that the subspaces spanned by $S=\{\alpha,\beta\}$ and $T=\{\alpha,\beta,\gamma\}$ are the same.

19. If a vector β of a vector space V(F), is linear combination of vectors $\alpha_1,\alpha_2,...,\alpha_n$, then show that the set of vectors $\beta,\alpha_1,\alpha_2,...,\alpha_n$ is linearly dependent.

20. Let S be a linearly independent subset of a vector space V. Suppose β is a vector which is not in the subspace spanned by S. Then show that the set obtained by adjoining β to S is linearly independent.

21. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s,2+5s)$ is not a vector space.

22. Let $W=\text{Span}\{v_1,v_2,\cdots,v_p\}$, where $v_1,v_2,\cdots,v_p$ are in a vector space V. Show that v_k is in W for $1\le k\le p$.

23. An $n\times n$ matrix A is said to be symmetric if $A^T=A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of 3×3 matrices.

24. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W=\left\{\begin{bmatrix}x\\y\end{bmatrix}: xy\ge 0\right\}$.

(a) If **u** is in W and c is any scalar, is c**u** in W ? Why?

(b) Find specific vectors **u** and **v** in W such that $u+v$ is not in W.

25. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H=\left\{\begin{bmatrix} x \\ y \end{bmatrix} : x^2+y^2 \le 1\right\}$. Show that H is not a subspace of $\mathbb{R}^2$.

26. In the following exercises, determine if the given set is a subspace of P_n, the set of polynomials with degree at most n, for an appropriate value of n. Justify your answers.

(a) All polynomials of the form $p(t)=at^2$, where a is in R.

(b) All polynomials of the form $p(t)=a+t^2$, where a is in R.

(c) All polynomials of degree at most 3, with integers as coefficients.

(d) All polynomials in P_n such that $p(0)=0$.

27. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector **v** in $\mathbb{R}^3$ such that $H=\text{Span}\{\text{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

28. Let W be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vectors **u** and **v** such that $W=\text{Span}\{\text{u,v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

29. Let W be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

30. Let $v_1=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3=\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $w=\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) How many vectors are in $\text{Span}\{v_1, v_2, v_3\}$?

(b) Is **w** in the subspace spanned by v_1, v_2, v_3? Why?

31. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $w = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is **w** in the subspace spanned by v_1, v_2, v_3? Why?

32. The set of all continuous real-valued functions defined on a closed interval [a, b] in R is denoted by C[a, b]. This set is a subspace of the vector space of all real-valued functions defined on [a, b].

(a) What facts about continuous functions should be proved in order to demonstrate that C[a, b] is indeed a subspace as claimed?

(b) Show that $\{f \in C[a,b] : f(a) = f(b)\}$ is a subspace of C[a, b].

33. For fixed positive integers m and n, the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

(a) Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

(b) Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.

34. In a vector space V, show that

(a) the zero vector is unique.

(b) the vector (–u) such that u + (–u) = 0 is unique.

(c) 0u = 0, for all u in V.

(d) c0 = 0, for all scalar c.

(e) (–1)u = – u, for all u in V.

(f) if cu = 0 for some non-zero scalar c, then u = 0.

35. Let **u** and **v** be vectors in a vector space V, then show that Span{u, v} is the smallest subspace of V that contains both **u** and **v**.

36. Let H and K be subspaces of a vector space V. Show that $H \cap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

37. Given subspaces H and K of a vector space V, the **sum** of H and K, written as H + K, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

H + K = {**w** : **w** = **u** + **v** for some **u** in H and some **v** in K}

(a) Show that H + K is a subspace of V.

(b) Show that H is a subspace of H + K and K is a subspace of H + K.

38. Suppose $u_1, u_2, \cdots, u_p$ and $v_1, v_2, \cdots, v_q$ are vectors in a vector space V, and let $H = \text{Span}\{u_1, u_2, \cdots, u_p\}$ and $K = \text{Span}\{v_1, v_2, \cdots, v_q\}$. Show that $H + K = \text{Span}\{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_q\}$.

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