

- (g) Suppose that a function  $f$  is analytic throughout a disc  $|z - z_0| < R_0$  centred at  $z_0$  and with radius  $R_0$ . Then prove that  $f(z)$  has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

$$\text{where } a_n = \frac{f^n(z_0)}{n!}, \quad (n = 0, 1, 2, \dots)$$

- (h) State and prove Laurent's theorem.

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3 (Sem-6/CBCS) MAT HC 1

2022

**MATHEMATICS**

(Honours)

Paper : MAT-HC-6016

**(Complex Analysis)**

Full Marks : 60

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer **any seven** questions from the following: 1×7=7
- (a) If  $c$  is any  $n$ th root of unity other than unity itself, then value of  $1 + c + c^2 + \dots + c^{n-1}$  is
- (i)  $2n\pi$
  - (ii) 0
  - (iii) -1
  - (iv) None of the above
- (Choose the correct answer)*

(b) The square roots of  $2i$  is

(i)  $\pm(1+i)$

(ii)  $\pm(1-i)$

(iii)  $\pm \frac{1}{\sqrt{2}}(1-i\sqrt{2})$

(iv) None of the above  
(Choose the correct answer)

(c) A composition of continuous function is

(i) discontinuous

(ii) itself continuous

(iii) pointwise continuous

(iv) None of the above  
(Choose the correct answer)

(d) The value of  $\text{Log}(-ei)$  is

(i)  $\frac{\pi}{2} - i$

(ii)  $i$

(iii)  $1 - \frac{\pi}{2}i$

(iv) None of the above  
(Choose the correct answer)

(e) The power expression of  $\cos z$  is

(i)  $\frac{e^z + e^{-z}}{2}$

(ii)  $\frac{e^{iz} + e^{-iz}}{2}$

(iii)  $\frac{e^{iz} + e^{-iz}}{2i}$

(iv) None of the above  
(Choose the correct answer)

(f) The Cauchy-Riemann equation for analytic function  $f(z) = u + iv$  is

(i)  $u_x = v_y, u_y = -v_x$

(ii)  $u_x = -v_y, u_y = v_x$

(iii)  $u_{xx} + v_{yy} = 0$

(iv) None of the above  
(Choose the correct answer)

(g) If  $w(t) = u(t) + iv(t)$ , then  $\frac{d}{dt}[w(t)]^2$  is equal to

(i)  $2[u(t) + iv(t)]$

(ii)  $2w'(t)$

(iii)  $2w(t)w'(t)$

(iv) None of the above  
(Choose the correct answer)

- (h) What is Laplace's equation ?  
 (i) What is extended complex plane ?  
 (j) What is Jordan arc ?

2. Answer **any four** questions from the following : 2×4=8

(a) Write principal value of  $\arg\left(\frac{i}{-1-i}\right)$ .

(b) If  $f(z) = x^2 + y^2 - 2y + i(2x - 2xy)$ , where  $z = x + iy$ , then write  $f(z)$  in terms of  $z$ .

(c) Use definition to show that  $\lim_{z \rightarrow z_0} \bar{z} = \bar{z}_0$ .

(d) Find the singular point of

$$f(z) = \frac{z^2 + 3}{(z+1)(z^2 + 5)}$$

(e) If  $f'(z) = 0$  everywhere in a domain  $D$ , then prove that  $f(z)$  must be constant throughout  $D$ .

(f) Evaluate  $f'(z)$  from definition, where  $f(z) = \frac{1}{z}$ .

(g) If  $f(z) = \frac{z}{\bar{z}}$ , find  $\lim_{z \rightarrow 0} f(z)$ , if it exists.

(h) Write the function  $f(z) = z + \frac{1}{z}$  ( $z \neq 0$ ) in the form  $f(z) = u(r, \theta) + iv(r, \theta)$ .

3. Answer **any three** questions from the following : 5×3=15

(a) If  $z_1$  and  $z_2$  are complex numbers, then show that  $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ .

(b) Show that  $\exp(2 \pm 3\pi i) = -e^2$ .

(c) Sketch the set  $|z - 2 + i| \leq 1$  and determine its domain.

(d) Let  $C$  be the arc of the circle  $|z| = 2$  from  $z = 2$  to  $z = 2i$ , that lies in the 1st quadrant, then show that

$$\left| \int_C \frac{z-2}{z^2+1} dz \right| \leq \frac{4\pi}{15}$$

(e) Evaluate  $\int_C \frac{dz}{z}$ , where  $C$  is the top half of the circle  $|z|=1$  from  $z=1$  to  $z=-1$ .

(f) If  $f(z)=e^z$ , then show that it is an analytic function.

(g) If  $f(z)=\frac{z+2}{z}$  and  $C$  is the semi circle  $z=2e^{i\theta}$ ,  $(0 \leq \theta \leq \pi)$ , then evaluate  $\int_C f(z) dz$ .

(h) Find all values of  $z$  such that  $e^z = -2$ .

4. Answer **any three** questions from the following: 10×3=30

(a) State and prove Cauchy-Riemann equations of an analytic function in polar form.

(b) Suppose that

$$f(z) = u(x, y) + iv(x, y), \quad (z = x + iy)$$

and  $z_0 = x_0 + iy_0$ ,  $w_0 = u_0 + iv_0$ , then

prove that if  $\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0$

and  $\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0$  then

$\lim_{z \rightarrow z_0} f(z) = w_0$  and conversely.

(c) If the function  $f(z) = u(x, y) + iv(x, y)$  is defined by means of the equation

$$f(z) = \begin{cases} \frac{\bar{z}^e}{z}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0, \end{cases}$$

then prove that its real and imaginary parts satisfies Cauchy-Riemann equations at  $z=0$ . Also show that  $f'(0)$  fails to exist.

(d) If the function

$$f(z) = u(x, y) + iv(x, y) \text{ and its}$$

conjugate  $\bar{f}(z) = u(x, y) - iv(x, y)$  are both analytic in a domain  $D$ , then show that  $f(z)$  must be constant throughout  $D$ .

(e) If  $f$  be analytic everywhere inside and on a simply closed contour  $C$ , taken in the positive sense and  $z_0$  is any point interior to  $C$ , then prove that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

(f) State and prove Liouville's theorem.